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HAMILTONIAN SYSTEMS WITH A SPECIFIED INVARIANT MANIFOLD AND SOME OF THEIR APPLICATIONS*

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The problem of constructing a Hamiltonian system whose phase flux leaves a specified manifold invariant is solved, and the corresponding, structurally simple Hamiltonian is found. The system is used to construct the solution of the Cauchy problem with an unknown boundary for a non-linear, first-order differential equation. A similar problem arises when studying singularities in the theory of optimal control and differential games /1-3/. The second application of the results obtained consists of the study of two types of singularities encountered in extremal problems of dynamics. The paper generalizes certain results obtained in [4].

1. Non-characteristic submanifolds. This paper investigates the geometry of submanifolds of odd codimensions in the contact manifold. These are then used to generalize the classical method of characteristics referring to the submanifolds of unit codimensions (hypersurfaces) [5] to make them applicable to the Cauchy problem with an unknown boundary for first-order equations.

The material of Sect.1—4 can be conveniently explained using the geometrical language of e.g. [5, 6]. Below, we shall use, in particular, the concept of a contact structure.

We shall give the name contact manifold to a pair (M^{2n+1}, β) where M^{2n+1} is a smooth $(2n+1)$-dimensional manifold and β is a differential 1-form on this manifold such, that the 2-form $d\beta$ is non-degenerate on the hyperplane $\beta = 0$ in the space T_zM tangent to M at any point $z \in M$. We shall neglect, on some occasions, the upper index denoting the dimensions of the manifold. A typical example of a contact manifold is the space R^{2n+1} whose points we shall write in the form (x, p, u) where $x, p \in R^n$, $u \in R$, with the differential form $\alpha = du - \Sigma p_i dx_i$.

Let $W^{2(n-k)} \subset M^{2n+1}$ be a submanifold of even dimenions $2(n-k), k \leqslant n$ (or, which amounts to the same thing, of odd dimensions $2k+1)$), in the contact manifold M. We shall call W a submanifold non-characteristic at the point $z \in W$, if

1) the submanifold W is transversal to the contact hyperplane, i.e. the intersection P_z of the plane T_zW tangent to W with the contact hyperplane $\beta = 0$ has dimensions $2(n-k)-1$;

2) the form $d\beta$ on P_z has rank $2(n-k-1)$, i.e. the kernel l_z of the form $d\beta$ on P_z is of unit dimensions.

We shall call the submanifold W non-characteristic if it is non-characteristic at all points. If $k=0$, i.e. W is a hypersurface in M, then the second condition follows from the first, and the definition given above becomes standard [5].

Clearly, the submanifold $W^{2(n-k)} \subset M^{2n+1}$ of the general position is (locally) non-characteristic.

The kernel l_z of the form $d\beta$ on $P_z \subset T_zW$ (see condition 2)) will be called the characteristic direction on W, and the integral curves of the field of characteristic directions are the characteristics of the manifold W. When $k=0$ the above definitions are the same as the standard definitions [5].

Below we shall show that the characteristic directions are obtained by restricting on W the directions of some Hamiltonian vector field on M, and we shall give explicit formulas for the corresponding Hamiltonian.

2. Hamiltonian systems. In the method of characteristics, we place the vector field ξ_Φ, in correspondence with the function Φ on the contact manifold (M, β), the field being defined by the following conditions:

$$\beta(\xi_\Phi) = 0;\ d\beta(\xi_\Phi, \eta) = d\Phi(\eta), \text{ if } \beta(\eta) = 0 \tag{2.1}$$

In the case of the manifold (R^{2n+1}, α) the field ξ_Φ has components $(\partial\Phi/\partial p, -\partial\Phi/\partial x - p\partial\Phi/\partial u, (p, \partial\Phi/\partial p))$. We shall call Φ a Hamiltonian, and ξ_Φ a Hamiltonian field with the Hamiltonian Φ. In a number of cases a different definition [6] is convenient. The field ξ_Φ touches the hypersurface $\Phi = 0$ and defines on it a characteristic direction. Using the Hamiltonian fields ξ_Φ we form the "Poisson's bracket" of smooth functions Ψ and Φ on M according to the formula

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$$\{\Psi\Phi\} = d\beta\,(\xi_\Phi,\ \xi_\Psi) = \xi_\Phi\Psi \tag{2.2}$$

where the expression on the right hand side is a derivative of Ψ in the direction ξ_Φ.

The quantity (2.2) can be written in the space (R^{2n+1}, α) in the form

$$\{\Psi\Phi\} = \sum_{i=1}^{n}\left(\frac{\partial\Psi}{\partial x_i} + \frac{\partial\Psi}{\partial u}p_i\right)\frac{\partial\Phi}{\partial p_i} - \left(\frac{\partial\Phi}{\partial x_i} + \frac{\partial\Phi}{\partial u}p_i\right)\frac{\partial\Psi}{\partial p_i} \tag{2.3}$$

and is also called the Jacobi bracket [7]. The term Poisson's bracket is used more often when Φ and Ψ are independent of u.

3. Characteristic directions.

Let $W^{2(n-k)} \subset M^{2n+1}$ be a submanifold of codimensions $2k+1$ in the contact manifold (M^{2n+1}, β), defined by the $(2k+1)$-th equation $F_i = 0$, $i = 0, 1, \ldots, 2k$, where F_i are smooth functions on M. The following proposition establishes the criterion of the non-characteristic nature of W.

Proposition 1. Let $z \in W$ and the forms $dF_i(z)$ and $\beta(z)$ be all independent. Then W is non-characteristic at the point z if and only if the matrix

$$A = \|a_{ij}\| = \|\{F_iF_j\}\|,\ i, j = 0, 1, \ldots, 2k \tag{3.1}$$

has (highest possible) rank of $2k$.

Indeed, if dF_i and β are independent forms, W is a submanifold and the first of the conditions of non-characteristic nature given in Sect.1 is satisfied.

Let us consider the contact hyperplane $\beta = 0$ in T_zM, denoting it by N, and denote by $P \subset N$ the intersection $N \cap T_zW$, by $L^c \subset N$, the subspace generated by the vectors $\xi_{F_i}(z)$, and by ω the bound of the form $d\beta$ on N. From the definition of the contact structure it follows that ω is a non-degenerate form, and the definition of the Hamiltonian vector fields implies that the spaces P and L^c are orthogonal supplements of each other relative to the form ω. Indeed, the condition $\omega(\xi_{F_i}(z), \eta) = 0$, $\eta \in N$ is equivalent to the fact that $dF_i(\eta) = 0$, $i = 0, 1, \ldots, 2k$, i.e. that η is a vector tangent to W. Further, since

$$d\beta(\xi_{F_i}, \xi_{F_j}) = dF_i(\xi_{F_j}) = \xi_{F_j}F_i = \{F_iF_j\}$$

it follows that the condition imposed on the rank of the matrix A will be reformulated thus: the kernel $\ker(\omega \mid L^c)$ of the form ω is one-dimensional in the space L^c. However $\ker(\omega \mid P) = P \cap L^c = \ker(\omega \mid L^c)$, since P and L^c supplement each other orthogonally. Consequently the conditions $\dim\ker(\omega \mid P) = 1$ and $\dim\ker(\omega \mid L^c) = 1$ are equivalent and the proposition is proved.

Corollary. The characteristic direction l_z on W coincides with the direction of the Hamiltonian vector field ξ_H, where $H = \theta_0F_0 + \theta_1F_1 + \ldots \theta_{2k}F_{2k}$, and the vector function $\theta = (\theta_0, \ldots, \theta_{2k})$ satisfies the equation

$$A\theta = 0 \tag{3.2}$$

where the matrix A is given by (3.1).

Indeed, in proving Proposition 1 we have established that $l_z = \ker(\omega \mid L^c)$, where L^c is a space stretched over the vectors $\xi_i = \xi_{F_i}(z)$. The kernel $\ker(\omega \mid L^c)$ of the form ω on L^c consists clearly of the vectors $\xi = \Sigma\theta_i\xi_i$, such that

$$\omega(\xi, \xi_i) = \Sigma\theta_j\omega(\xi_j, \xi_i) = \Sigma a_{ij}\theta_j = 0,\quad i = 0, \ldots, 2k.$$

which proves the corollary.

The solution of the system (3.2) can be obtained in explicit form. Let us fix the index j and put in (3.2) $\theta_i = \Delta_{ij}$ where Δ_{ij} is the cofactor of the element a_{ij} of A. Using Cramer's rule, we obtain

$$\sum_{i=0}^{2k} a_{mi}\Delta_{ij} = \delta_{mj}\det A = 0,\quad m, j = 0, 1, \ldots, 2k$$

since A is a skew symmetric matrix of odd order. Thus system (3.2) has a solution θ with components $\theta_i = \Delta_{ij}$, representing the homogeneous forms of degree $2k$ of the elements of the matrix A. The conditions $\theta \neq 0$ and $\operatorname{rank} A = 2k$ are equivalent.

We shall show that system (3.2) has a solution $\theta = \lambda = (\lambda_0, \ldots, \lambda_{2k})$, where λ_i are forms of degree k of the elements of A. Let A_i be a metrix obtained from A by deleting the i-th row and i-th column. The matrix is skew symmetric, of order $2k$. If $B = \|b_{ij}\|$ represents any such matrix, then its Pfaffian is well defined [8]

$$\operatorname{Pf}(B) = \frac{1}{2^k k!}\sum_{j_s}\varepsilon(j_1, j_2, \ldots, j_{2k})\, b_{j_1j_2}\ldots b_{j_{2k-1}j_{2k}}$$

where $\varepsilon(j_1, \ldots, j_{2k})$ is the evenness of the subustitition $s \to j_s$. We know [8] that $[\operatorname{Pf}(B)]^2 = \det B$.

The following assertion generalizes this identity in a specific sense, and defines the required solution λ of system (3.2).

Proposition 2. Let A be a skew symmetric matrix of odd order. Then

1) $\Delta_{ij} = \lambda_i\lambda_j$, $\lambda_s = (-1)^s \operatorname{Pf}(A_s)$, $s = 0, 1, \ldots, 2k$;

2) $\Delta_{ij}^2 = \Delta_{ii}\Delta_{jj}$, i.e. Δ_{ij} is the geometric mean of Δ_{ii}, Δ_{jj};

3) the vector $\lambda = (\lambda_0, \ldots, \lambda_{2k})$ satisfies the equation $A\lambda = 0$.

The condition rank $A = 2k$ is equivalent to $\lambda \neq 0$.

To prove it, consider the matrix $A^\circ = \| a^\circ_{ij} \|$ of order $2k+2$, where $a^\circ_{ij} = a_{ij}$, of $0 \leqslant i, j \leqslant 2k$, $\overset{\circ}{a}_{i\, 2k+1} = v_i$, $\overset{\circ}{a}_{2k+1\, i} = -v_i$ if $0 \leqslant i \leqslant 2k$, $\overset{\circ}{a}_{2k+1\, 2k+1} = 0$, and v_i are independent variables. Then A° is a skew symmetric matrix of even order and we have

$$\det A^\circ = [\operatorname{Pf}(A^\circ)]^2 \tag{3.3}$$

It can be shown that

$$\det A^\circ = \sum_{i,j=0}^{2k} \Delta_{ij} v_i v_j, \quad \operatorname{Pf}(A^\circ) = \sum_{i=0}^{2k} (-1)^i \operatorname{Pf}(A_i)\, v_i$$

Therefore, assertion 1) of Proposition 2 is equivalent to (3.3). Assertion 2) obviously follows from 1), since $\Delta_{ii} = [\operatorname{Pf}(A_i)]^2$.

Earlier it was shown that $A\theta = 0$, where $\theta_i = \Delta_{ij} = \lambda_i\lambda_j$. Consequently $\lambda_j A\lambda = A\theta = 0$, and hence $A\lambda = 0$, which completes the proof.

Theorem 1. Let the submanifold $W^{2(n-k)} \subset M^{2n+1}$ be defined by the equations $F_i = 0$, $i = 0, 1, \ldots, 2k$ and let the condition of the non-characteristic nature hold: the forms dF_i and β are independent on W and the matrix $A = \| \{F_i F_j\} \|$ is of rank $2k$. Then the Hamiltonian

$$H = \sum_{i=0}^{2k} \lambda_i F_i, \quad \lambda_i = (-1)^i \operatorname{Pf}(A_i), \quad i = 0, \ldots, 2k \tag{3.4}$$

defines the characteristic vector field ξ_H on W.

We note that if the manifold $W \subset M$ is defined by the equations $F_i = 0$ only locally, then the assertion of the theorem is also of that character.

4. The Cauchy problem with unknown boundary. Consider the following problem. The functions $F_i(z) \in C^2$, $z = (x, p, u) \in R^{2n+1}$, $i = 0, 1, \ldots, 2k$, $k \leqslant n$, the manifold $\Gamma_{k+1} \subset R^n$ of codimensions $k+1$ and the integral manifold $L_{k+1} \subset W^{2(n-k)} = \{z \in R^{2n+1}\colon F_i(z) = 0,\ i = 0, 1, \ldots, 2k\}$, lying above Γ_{k+1} (i.e. the form $\alpha = 0$ on the planes tangent to L_{k+1} and the projection $\pi\colon z \to x$ define a **homeomorphism** of L_{k+1} and Γ_{k+1}) are all given. The functions F_i and the manifolds L_{k+1}, Γ_{k+1} are studied in the neighbourhood of the points $z^* = (x^*, p^*, u^*) \in R^{2n+1}$, $x^* \in R^n$. We require to find, in the neighbourhood of the point x^*, the function $u(x) \in C^2$ and smooth manifolds $\Gamma_j \subset R^n$, $j = 0, 1, \ldots, k$ of codimension j, satisfying the conditions

$$\Gamma_{k+1} \subset \Gamma_k \subset \ldots \subset \Gamma_0 \tag{4.1}$$

$$(x,\ \partial u(x)/\partial x,\ u(x)) \in L_{k+1}, \quad x \in \Gamma_{k+1} \tag{4.2}$$

$$F_i(x, \partial u(x)/\partial x, u(x)) = 0, \quad x \in \Gamma_j, \quad 0 \leqslant i \leqslant 2j;\ j = 0, \ldots, k \tag{4.3}$$

Conditions (4.2) plays the part of the boundary values in the Cauchy problem; equations (4.3) impose additional conditions on the unknown manifolds Γ_j, $j = 1, \ldots, k$; and $F_0 = 0$ holds in the neighbourhood Γ_0 of the point x^*.

We introduce the following notation: A^l_m is a square matrix of order $2l$, obtained by deleting the m-th row and column from the matrix $A^l = \| \{F_i F_j\} \|$, $0 \leqslant i, j \leqslant 2l$; $y_l \in R^n$ is a vector defined by the formula

$$y_l = \sum_{i=0}^{2l} (-1)^i \operatorname{Pf}(A_i^l) F_{ip}(x, p, u); \quad l = 1, \ldots, k, \quad y_0 = F_{0p}$$

The following assertion gives the sufficient condition of existence and uniqueness in the small of the solution of problem (4.1)—(4.3).

Theorem 2. Let the vectors $y_0, y_1, \ldots, y_k$ at the point z^* and the space T_x tangent to Γ_{k+1} at the point x^*, together generate R^n. Then a solution of the problem (4.1), (4.2) exists, is unique, and can be constructed by integrating the $(k+1)$-th system of Hamiltonian equations with Hamiltonians of the form (3.4)

$$H_l = \sum_{i=0}^{2l} (-1)^i \operatorname{Pf}(A_i^l) F_i, \quad l = 1, \ldots, k; \quad H_0 = F_0$$

To prove the theorem, we shall show that the initial problem can be reduced to that of solving a similar problem with the initial fixed manifold Γ_k of codimensions k. Then k steps will yield the integral submanifold $L_0 \subset W^{2n} = \{z \in R^{2n+1}: F_0(z) = 0\}$ lying above $\Gamma_0 \subset R^n$.

Since the projection $\pi: z \to x$ from L_0 to $\Gamma_0 \subset R^n$ is a diffeomorphism, we can write $L_0 = \{(x, p(x), u(x)): x \in \Gamma_0\}$ for some functions $p: R^n \to R^n$, $u: R^n \to R$. From the condition that $\alpha = 0$ on the tangent planes in L_0 we find [5, 6], that $p(x) = \partial u(x)/\partial x$; and we can now confirm that the function $u(x)$ and the manifolds $\Gamma_j = \pi(L_j)$, constructed in consecutive steps for $j = k$, $k-1, \ldots, 0$, represent a solution of the problem in question.

Thus it is sufficient to describe the construction of the integral submanifold $L_k \subset W^{2(n-k)}$, containing L_{k+1} and projecting itself diffeomorphically on $\Gamma_k = \pi(L_k)$. From the conditions of the theorem it follows that the characteristic directions on $W^{2(n-k)}$ are transversal to the integral submanifold $L_{k+1} \subset W^{2(n-k)}$ and or any $z \in L_{k+1}$ the plane T_z is stretched over $T_z L_{k+1}$, while the characteristic direction l_z projects isomorphically on $\pi(T_z)$. The field ξ_H (see Sect.2) where the Hamiltonian $H = H_k$, defines the characteristic direction on $W^{2(n-k)}$.

We know [5, 6] that the dimensions of the isotropic subsapce does not exceed half the dimension of the space with non-degenerate symplectic form, therefore, every space $T_z L_k$ tangent to L_k must contain a characteristic vector $\xi_H(z)$ otherwise we obtain an isotropic subspace $T_z L_k$ of dimensions $n-k$ in the non-degenerate symplectic space $(T_z W \cap \{\beta = 0\})/l_z$ of dimensions $2(n-k-1)$. Consequently L_k must be invariant with respect to the phase flux of the field ξ_H, i.e. it must consist of integral curves of the field ξ_H, emitted from L_{k+1}. Thus we arrive at the integral manifold $L_k \subset W^{2(n-k)}$, lying above $\Gamma_k = \pi(L_k) \subset R^n$ [5].

The procedure described above gives a means for constructing a solution of the proposed Cauchy problem with unknown boundary, and contains the proof of its uniqueness.

5. On the construction of singular manifolds in extremal problems of dynamics. We shall describe the procedure for constructing singular motions and manifolds, using the results of Sect.1–4. The procedure is substantiated in [4, 9] and in the present paper. We note that to use the approach proposed we must previously define the concept of the motion (especially of the singular motion) of a positionally controlled dynamic system. A method given in, e.g. [2], can be used to achieve this.

The basic unknown quantity in the problems of optimal control and differential games is a a function of the optimal result $V(x)$, $x \in R^n$ satisfying, at the points of smoothness, the Hamilton-Jacobi, Bellman-Isaacs equation $F_0(x, p) = 0$, $p = V_x$ [1]. We shall call the manifolds in R^n at whose points at least one of the quantities F_{0p}, p, V is discontinuous, the singular manifolds. The manifolds can contain segments of optimal motions, and these will also be called singular. We shall further assume that the manifold in question can be locally included in the closure of a region in which the relation $p = p(x)$ is smooth and continuously continuable to the closure. The values of x and $p(x)$ at the point x of a similar manifold $\Gamma \subset R^n$ can be connected by conditions of the form $F_i(x, p, V) = 0$, The latter have the meaning of the necessary conditions of optimality (see [9] and Sect.7 of the present paper). Moreover, the fundamental equation $F_0 = 0$ also holds on Γ. In the cases considered the total number of independent constraints (satisfying the condition of non-characteristic nature of Sect.3) $F_i = 0$ is odd $i = 0, 1, \ldots, 2k$, and k is equal to the codimensions of the manifold Γ. The manifold Γ itself is a projection on R^n of a family of solutions of the system

$$x^{\cdot} = H_p, \quad p^{\cdot} = -H_x - H_V p, \quad V^{\cdot} = (p, H_p) \tag{5.1}$$

released from a manifold $L_{k+1} \subset W$. The Hamiltonian of system (5.1) of the form (3.4) is constructed using the constraint function F_i as the basis. The manifold L_{k+1} must be determined specially in each specific problem [10].

Generally speaking, the set of functions F_i is defined non-uniquely. A set of functions F_i for which the vector $H_p = \sum \lambda_i F_{ip}$ determines the field of tangents to the singular motion (the trajectories) is of interest. The convenience in this case lies in the fact that the singular motions and the singular manifold Γ are determined simultaneously. The functions F_i

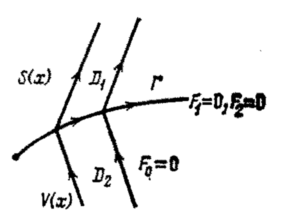

Fig.1

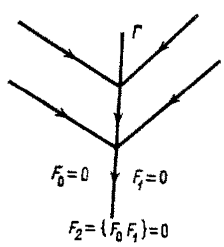

Fig.2

given in [9, 10] determine a field of vectors tangents to the singular motion.

Let us write the Hamiltonian (3.4) for $k=1$

$$H=\{F_0F_1\}F_2+\{F_2F_0\}F_1+\{F_1F_2\}F_0, \tag{5.2}$$

We assume that the function F_2 has a special form

$$F_2=\{F_0F_1\} \tag{5.3}$$

Let a function $F^*(x, p, V)$ vanish on the manifold $L_1\subset W^{2(n-1)}$, defined by the set F_0, F_1, F_2. Then F^* is the first integral of the system (5.1), (5.2). Taking (5.3) into account, we have on L_1

$$\{F_2F_0\}\{F^*F_1\}+\{F_1F_2\}\{F^*F_0\}=0 \tag{5.4}$$

Writing the system (5.1) for two sets of functions F_0, F_1, F_2 and F_0, F_1, F^* and taking (5.4) into account, we can confirm that in both cases the systems are identical apart from the differentiation parameter.

Thus condition (5.3) guarantees that when the function F_2 of the form (5.3) is replaced by any other function F^* vanishing on L_1, then the Hamiltonian field does not change and consequently the family of integral curves (motions) on Γ also remains unchanged.

Let us consider two types of singular manifolds of codimensions 1, which can be constructed using the Hamiltonian of the form (5.2).

6. Equivocal manifold. This manifold is encountered in problems of the theory of differential games [1] and has no analogue in optimal control. Quantitatively speaking, the equivocal manifold is a surface of refraction of the optimal trajectories (the switch-over of the controls of both players) along which singular optimal motions are also possible [9] (Fig.1).

When a similar manifold is constructed in the usual manner, the function $V(x)$ can be assumed given in some primary subregion D_1 where it will be denoted by $S(x)$. The function of the optimal result is continuous on the equivocal manifold

$$F_1(x, V)\equiv V-S(x)=0, \quad x\in\Gamma \tag{6.1}$$

and its gradient, satisfying the condition of optimality of the form $(q=S_x)$ [9]

$$F_2(x, p)\equiv G(x, q(x), p)=0 \tag{6.2}$$

is discontinuous.

The relation $G(x, q, p)=0$ in (6.2) resembles the Weierstrass-Erdman conditions in the variational calculus, in connecting the derivatives of the unknown function on both sides of the discontinuity, and makes possible the continuous matching of two smooth segments of the solution of the extremal problem.

Mathematically, the construction of the equivocal surface reduces to the Cauchy problem with an unknown boundary [4]. The solution of this problem yields the procedure for constructing the manifold Γ given in Sect.1–5. Here $k=1$, the function F_0 determines the basic equation and the function F_1, F_2 are given by the formuals (6.1), (6.2).

In [9] it was shown that two types of equivocal manifolds are possible. In the first case the optimal control (limit control from the region D_2) of one player is determined on the manifolds non-uniquely; the equations $F_2\equiv G=0$ and $F_0=0$ determine the optimality of two values of the control vector. In the second case the optimal controls are unique everywhere and the optimal motions must arrive at Γ from D_2 with tangential contact. The conditions of tangential contact are written in the form [9] (see conditions (5.3))

$$F_2(x, p)\equiv(F_{0p}, p-q(x))\equiv\{F_1F_0\}=0 \tag{6.3}$$

The equivocal surface which is an envelope, appears in the game theoretic problem of the pursuit of an inertialess object by the pursuer, in the presence of an obstacle within which both players are forbidden to appear [9, 10]. The Bellman-Isaacs equation in this case has the form $F_0\equiv F(p)+1=0$ where $F(p)$ is a homogeneous, first degree function. The initial solution $S(x)$ is equal to the time of pursuit from the initial position x, with the players moving along a geodesic. System (5.1) can in this case be reduced to the form [7, 8]

$$x^{\cdot}=F_p, \quad p^{\cdot}=[(S_{xx}F_p, F_p)/(F_{pp}q, q)](p-q) \tag{6.4}$$

where S_{xx}, F_{pp} are Hessians. The motions defined by the system (6.4) are curvilinear, and all other optimal motions in this problem (except the motion along the edge of the obstacle) are rectilinear. The case of objects moving in a plane $x\in R^4$, was studied in detail in [10].

7. Universal manifold. The universal hypersurface consists of the singular optimal trajectories. The optimal motions touch the trajectories on both sides of the surface (Fig.2). The function of optimal result $V(x)$ may be smooth on the universal manifold [11], and here we assume this to be the case. The maximum function (minimax for the game theoretic problems) defining the basic equation is discontinuous. Denoting this function by different symbols on each side of the surface, we arrive at the conditions

$$F_0(x, p) = 0, \quad F_1(x, p) = 0, \quad x \in \Gamma \; (p = V_x) \tag{7.1}$$

In addition to (7.1), we shall require that the following relation (see condition (5.3)) holds on Γ:

$$F_2(x, p) \equiv \{F_0 F_1\} = 0, \quad x \in \Gamma \tag{7.2}$$

When the function $V(x)$ is doubly smooth, condition (7.2) follows from (7.1). Indeed, differentiating equations (7.1) with respect to the component x_i of the vector x, we obtain

$$c_{mi} = \frac{\partial F_m}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial F_m}{\partial p_j} \frac{\partial p_j}{\partial x_i} = 0, \quad m = 0, 1; \; i = 1, \ldots, n$$

where p_j are the components of the vector p. It is clear that $c_{0i}\partial F_1/\partial p_i - c_{1i}\partial F_0/\partial p_i = 0$. Summing the left hand side from $i = 1$ to $i = n$ and taking into account the fact that $\partial p_j/\partial x_i = \partial p_i/\partial x_j$, we arrive at (7.2).

In the case of the problem of optimal control (e.g. with respect to high speed response) with Hamiltonian $H = H_0 + uH_1 - 1$, $|u| \leqslant 1$; $H_i = H_i(x, \psi)$, $\psi = -p$ linear with respect to the scalar control u, condition (7.2) represents the known necessary condition for the singular trajectory [3]. Indeed, the functions F_i have the form $F_0 = H_0 - H_1 + 1$, $F_1 = H_0 + H_1 + 1$, $H_i = H_i(x, p)$. From this we obtain, on the singular motion

$$H_1 \equiv 0, \quad H_1^{\cdot} = {}^1/_2 \{F_0 F_1\} = \{H_1 H_0\} = 0.$$

Let us assume that the functions (7.1), (7.2) are independent and use the Hamiltonian (5.2) to construct the system $x^{\cdot} = H_p$, $p^{\cdot} = -H_x$ on the manifold W

$$\begin{aligned} x^{\cdot} &= \{\{F_0 F_1\} F_1\} F_{0p} + \{F_0 \{F_0 F_1\}\} F_{1p} \\ p^{\cdot} &= -\{\{F_0 F_1\} F_1\} F_{0x} - \{F_0 \{F_0 F_1\}\} F_{1x} \end{aligned} \tag{7.3}$$

System (7.3) represents the description of the well-known equations of singular characteristics [3] in the form of a sliding mode. More accurately, the sliding mode corresponds to a choice of the differentiation parameter in (7.3) such that the sum of the coefficients accompanying F_{0p} and F_{1p} is equal to unity.

The proposed approach enables us to write, in the cases in question, the equations for the motions lying on the discontinuity manifold using the normal procedure of constructing the canonical equations with a smooth Hamiltonian. the control parameters are eliminated from the process after the corresponding extrema have been computed. The programmed optimal control for the singular motion can be recovered after constructing the solution for the canonical system, using its right hand sides as the resulting control for the sliding mode. Here the relative proportion of utilisation of the phase velocity F_{ip} in the sliding mode is proportional to the λ_i-form of order k relative to the Poisson brackets assembled from the functions F_j, $j = 0, 1, \ldots, 2k$.

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